МАТЕМАТИЧЕСКИЙ АНАЛИЗ ЭКОНОМИЧЕСКИХ МОДЕЛЕЙ

Predicting the trajectory of economic cycles

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Abstract. This paper deals with the development of a method for predicting the trajectory of a pseudo-stationary fragment of the economic cycle. The latter is represented by discrete values (readouts) of random oscillations of the income function. The statistical equivalence of these readouts and second-order autoregression (Yule series) led to the adaptation of the autoregressive model to the specified fragment of the cycle. It is proposed to use the adapted autoregressive model as a tool for predicting cycle values via the method of statistical tests (Monte-Carlo) by forming the most probable cycle trajectory. The procedure for the formation of the cycle trajectory is described in detail and its parameters have formal justifications. The content of the subsequent statistical analysis of the simulation results is illustrated by the example of determining the instant of the predicted peak value of the cycle. The presented method is applicable in macroeconomic and econometric problems, the solution to which requires knowledge of the predicted trajectory of the cycle under consideration.

Keywords: economic cycle, random oscillations, Yule series, maximum likelihood estimates, pseudo-stationarity, cycle trajectory.

JEL Classification: C02, C15, C22.

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INTRODUCTION

Economy of a specified territory (region, country, world) is a mega-system with very slow processes that can last for years or even decades. Human participation in the implementation of these processes makes it possible to influence the trajectory of economic development in real time. For the conscious management of economic processes, it is necessary to know the mechanism of its formation. Knowledge about such a mechanism is formalized, as a rule, in the form of its mathematical model.

The model of economic cycles, as proposed in reference (Karmalita, 2020), is based on a probabilistic description of the investment function and the perception of the economic system as a material object with certain inherent properties. In accordance with this approach, the investment function $I(t)$ is the sum of all ($N < \infty$) existing investments, each of which is represented in the form shown in Fig. 1.

Here C_j, ΔC_j, and T_j are the initial capital, return and duration of the j^{th} investment cycle. Formally, the function $I(t)$ is the sum with two terms:

$$I(t) = \sum_{j=1}^{N} I_j(t) = M(t) + E(t), \quad (1)$$

where $M(t)$ is the deterministic component of the investment function, and $E(t)$ is the stochastic one. This means that $M(t)$ forms a long-term trend of income, whose oscillations $\Xi(t)$ are induced by fluctuations $E(t)$ in investments.

As for the system model itself, it represents the cycle as random oscillations generated by a linear elastic system with the natural frequency $f_0 = 1/T_0$ and damping factor h under the influence of the Gaussian white noise $E(t)$ (Bolotin, 1984). Mathematically, the cycle model is represented by an ordinary differential equation of the second order:

$$\ddot{\Xi}(t) + 2h\dot{\Xi}(t) + (2\pi f_0)^2 \Xi(t) = E(t), \quad (2)$$

where $\Xi(t)$ represents random oscillations of income, and $E(t)$ — fluctuations of investments in the form of white noise. This model provides the ability to manage a cycle trajectory via spurring/curbing of only those investments that have durations correlated to the cycle period T_0 (Karmalita, 2020). To clarify this approach,

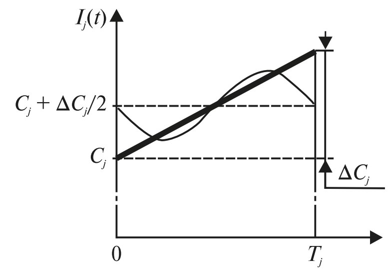

Fig. 1. Diagram of the j^{th} investment cycle

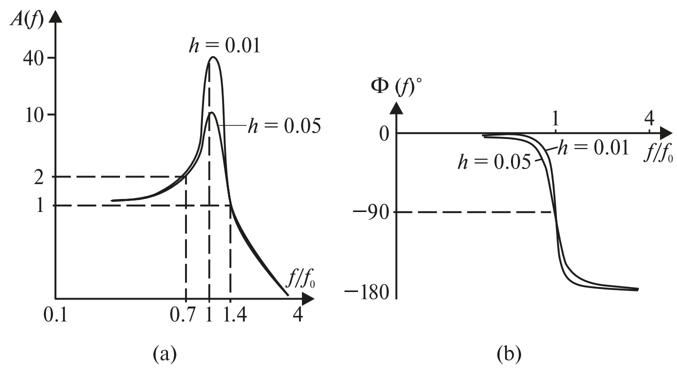

Fig. 2. Amplitude and phase frequency characteristics of the linear elastic system

let us turn to the variable values of the investment cycle (Fig. 1), which correspond to the period of the so-called sawtooth function $F(t)$. This periodic function can be represented by the infinite Fourier series:

$$F(t) = \frac{\Delta C_j}{\pi} \sum_{l=1}^{\infty} \sin(2\pi l t / T_j) / l.$$

In particular, the first ($l = 1$) harmonic $\sin(2\pi t/T_j)$ is shown in Fig. 1. This means that the concept of the frequency (period) of the economic cycle is quite applicable to the analysis of economic systems. Therefore, consider the properties of the linear elastic system in the frequency domain, which are described by its amplitude $A(f)$ and phase $\Phi(f)$ frequency characteristics (Fig. 2).

$A(f)$ determines the ratio of the amplitudes of the input and output harmonics of the system. $\Phi(f)$ is the difference between their phases, which is equivalent to the time delay of the output process with respect to the input process.

Fig. 2a clearly demonstrates that the main part of the range ($\pm 3\sigma_\xi$) of income oscillations is formed by investment fluctuations concentrated in the frequency range $0.7 \leq f/f_0 \leq 1.4$. This fact allows targeted management of the cycle affecting only investments with the following durations T_j: $T_0 / 1.4 \leq T_j \leq T_0 / 0.7$.

From Fig. 2b it follows that these actions lead to a tangible change in the intensity of the considered cycle $\Xi(t)$ with a time delay of a quarter of the cycle period ($T_0/4$). Therefore, the practical application of the above approach assumes advanced (at least, by half cycle period) knowledge of moment t_p of the cycle peak. This fact determines the purpose of this article — the development of a method for predicting the trajectory of the economic cycle.

PREDICTING THE TIME OF THE CYCLE PEAK

We'll proceed from the fact that there is a fragment of the income function $X(t)$ on the interval $t_0, \ldots, t_c$, where t_c is the current moment of time. Moreover, this fragment is assumed to be pseudo-stationary (Karmalita, 2020), that is, the evolutionary change in the parameters of model (2) over indicated time interval corresponds to the statistical variability of their estimates with a confidence level P. Further, we assume that the fragment has a discrete representation with a sampling interval Δt: $x_i = X(t_i) = X(i\Delta t)$, $i = 1, \ldots, n$.

Recall that values ξ_i of the business cycle may be formed from readouts x_i of the income function by means of a filter with the bandwidth $f_1(0.7 f_0), \ldots, f_2(1.4 f_0)$: $\xi_i = \sum_{l=1}^{q} c_l x_{i-l}$.

In reference (Karmalita, 2020), a discrete model of random oscillations $\Xi(t)$ is proposed in the form of the second-order autoregressive model AR(2):

$$\xi_i = a_1 \xi_{i-1} + a_2 \xi_{i-2} + \varepsilon_i. \quad (3)$$

The random process generated by model (3) is called the Yule series. From the condition of its statistical equivalence to the random oscillation readouts, the relationship between the parameters of model (2) and the coefficients of model (3) was established in the form of the following expressions (Karmalita, 2020):

$$h = -0.5 \ln(-a_2) / \Delta t; \quad f_0 \approx (2\pi \Delta t)^{-1} \cos^{-1}\left(a_1 / 2\sqrt{-a_2}\right). \quad (4)$$

Thus, the problem of predicting the trajectory of the economic cycle is reduced to calculating the subsequent values of the AR(2) process. The initial values of these calculations are readouts ξ_{n-1} and ξ_n from the pseudo-stationary fragment of the concerned cycle.

To implement the procedure for predicting the cycle trajectory, it is necessary to adapt model (3) to the empiric values of ξ_i ($i = 1, ..., n$). Recall that for the first two correlations of the Yule series, it is possible to compose the Yule–Walker system of equations (Box et al., 2015): $\rho_1 = a_1 + a_2\rho_1$; $\rho_2 = a_1\rho_1 + a_2$.

This system of equations can be represented in the matrix form as:

$$\mathbf{B}\times\mathbf{A} = \mathbf{P}, \tag{5}$$

where

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix}.$$

Implementation of the Cramer's rule for solving this system yields a relationship of Yule model factors with series correlations: $a_1 = \rho_1(1-\rho_2)/(1-\rho_1^2)$, $a_2 = (\rho_2 - \rho_1^2)/(1-\rho_1^2)$.

Therefore, the use of maximum likelihood estimates (*MLE*s) for covariances and correlations of random oscillation ξ_i in the forms (Brandt, 2014): $\tilde{\gamma}_k = \frac{1}{n-k}\sum_{i=1}^{n-k}\xi_i\xi_{i+k}$, and $\tilde{\rho}_k = \tilde{\gamma}_k / \tilde{\gamma}_0$ provides the following *MLE*s of the coefficients of the AR(2) model:

$$\tilde{a}_1 = \frac{\tilde{\rho}_1(1-\tilde{\rho}_2)}{1-\tilde{\rho}_1^2}, \ \tilde{a}_2 = \frac{\tilde{\rho}_2 - \tilde{\rho}_1^2}{1-\tilde{\rho}_1^2}. \tag{6}$$

Hereinafter, the symbol «~» denotes the estimates of the corresponding parameters (coefficients). The presence of estimates (6) allows, using expression (4), to estimate the frequency f_0:

$$\tilde{f}_0 = 1/\tilde{T}_0 \approx (2\pi\Delta t)^{-1}\cos^{-1}\left(0.5\tilde{a}_1/\sqrt{-\tilde{a}_2}\right).$$

The adapted model $\xi_j = \tilde{a}_1\xi_{j-1} + \tilde{a}_2\xi_{j-2} + \varepsilon_j$ makes it possible to form the cycle realization $\xi_{n+1},\ldots,\xi_{n+m}$ ($m = 1.5\tilde{T}_0/\Delta t$). The stochastic nature of investment fluctuations $E(t)$ predetermines the implementation of this procedure via the method of statistical tests, in which ε_j is simulated by random numbers. In other words, modeling of the cycle trajectory can be carried out by the Monte Carlo method (Mazhdrakov et al., 2018). To utilize this method, one can use a pseudo-random number generator (*PRNG*) which is available in the software of modern computers. The numbers thus formed are called pseudo-random, since they are only approximations of true random numbers, at least because they have a period of repetition of their values.

The essence of statistical tests is as follows. Operator AR(2) with estimates $\tilde{a}_1$ and $\tilde{a}_2$ can be considered as a model of a corresponding elastic system with input ε_j and output ξ_j (Fig. 3).

In other words, the processing of values ξ_{j-1}, ξ_{j-2}, and ε_j by the operator AR(2) yields the value ξ_j corresponding to the time instant t_j.

Recall that the random numbers ε_j have a Gaussian distribution with a zero mathematical expectation and the root-mean-square (*rms*) value $\sigma_\varepsilon = \sigma_\xi/K_\sigma$, where K_σ is the *rms* gain of the system (Karmalita, 2020). Considering the above expression for estimate $\tilde{\gamma}_{k=0} = \tilde{\sigma}_\xi^2$, as well as the expression for $\tilde{K}_\sigma$ in the form $\tilde{K}_\sigma = \sqrt{(1-\tilde{a}_2)/\{(1+\tilde{a}_2)[(1-\tilde{a}_2)^2 - \tilde{a}_1^2]\}}$, we can write $\tilde{\sigma}_\varepsilon = \tilde{\sigma}_\xi/\tilde{K}_\sigma = \sqrt{\sum_{i=1}^{n}\xi_i^2/n}/\tilde{K}_\sigma$.

Therefore, m-fold repetition the calculations of the value ξ_j forms the predicted income oscillations $\xi_{n+1},\ldots,\xi_{n+m}$. As an example, Fig. 4 shows the trajectory of the process $\xi_j = 1.47\xi_{j-1} - 0.83\xi_{j-2} + \varepsilon_j$, which is represented by 10 readouts per period T_0 of the simulated cycle.

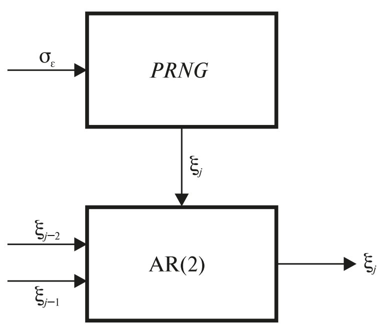

Fig. 3. Statistical simulation of the future cycle values

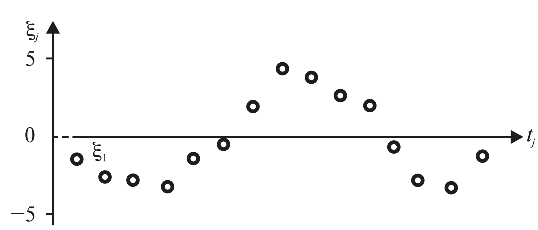

Fig. 4. Readouts of the predicted cycle trajectory

Due to the randomness of ξ_j values, the Monte-Carlo method should provide a representative set of ξ_{jl} (l = 1, ..., k) for each time t_j. Determining the predicted cycle peak is a statistical inference (Zacks, 1981), which is usually made by analyzing the properties of some statistics calculated from the values of the empirical data in question. In our case, such statistics will be the average value of simulated readouts ξ_{jl}:

$$\bar{\xi}_j = \sum_{l=1}^{k} \xi_{jl} / k. \quad (7)$$

As an example, consider the value $\xi_1 = -1.32$ shown in Fig. 4. During statistical tests with $k = 15$, the readouts ξ_{1l} were in the range -2.62, ..., 0.08, and their average value was $\bar{\xi}_1 = -1.04$. Hence, it becomes necessary to correctly choose the number (k) of generated realizations $\xi_{(n+1)\,l}$, ..., $\xi_{(n+\mathrm{m})l}$.

We can make this choice based on the following condition. Let the scatter of the statistics $\bar{\xi}_j$ not exceed the first order of smallness of σ_ξ, that is, $\sigma_{\bar{\xi}_j} < \sigma_\xi / 10$. Then, considering that *rms* values of variables in expression (7) are linked via $\sqrt{k}$ (Brandt, 2014), we get the value of $k > 100$.

The realization of estimates $\bar{\xi}_j$ obtained from the results of statistical tests makes it possible to determine the parameters of the predicted trajectory. For example, the moment of time t_p is found by enumerating the values $\bar{\xi}_j$ until the largest one is obtained.

CONCLUSIONS

This paper presents a statistical approach for predicting the trajectory of a pseudo-stationary fragment of the economic cycle, represented by readouts of income oscillations. It is based on the use of second order autoregression to simulate the cycle trajectory. The autoregressive model adapted to the specified cycle fragment turned out to be an effective tool for predicting cycle values. The method of statistical tests (Monte-Carlo) was used for the model's implementation. The corresponding procedure for the formation of the cycle trajectory is described in detail. The proposed rules for choosing the parameters of this procedure have formal justification. The subsequent statistical analysis of the simulation results has also been established. As the statistics $\bar{\xi}_j$ is an estimate of the mathematical expectation (mode) of the Gaussian distribution, the generated realization of estimates $\bar{\xi}_j$ can be classified as the most probable trajectory of the cycle. Accordingly, the parameters of the trajectory, for example, the moment of its peak (t_p), will also be the most likely.

The developed method is applicable in macroeconomic and econometric problems, the solution of which requires knowledge of the predicted trajectory of the cycle under consideration.

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Прогнозирование траектории экономических циклов

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Аннотация. Статья посвящена разработке метода прогноза траектории псевдостационарного фрагмента экономического цикла, представленного дискретными отсчётами случайных колебаний функции доходов. Статистическая эквивалентность последних процессу авторегрессии второго порядка (ряд Юла) обусловила применение модели этого ряда для прогноза траектории цикла. Реализация этой процедуры осуществляется методом статистических испытаний (Монте-Карло) с целью формирования наиболее вероятной траектории цикла. Определены как формальные параметры этих испытаний, так и содержание последующего статистического анализа результатов моделирования. Представленный в работе подход иллюстрируется примером определения момента наступления прогнозируемого пикового значения цикла. Разработанный метод применим в макроэкономических и эконометрических задачах, требующих знания прогнозируемой траектории рассматриваемого цикла.

Ключевые слова: экономический цикл, случайные колебания, ряд Юла, оценки максимального правдоподобия, псевдостационарность, траектория цикла.

Классификация JEL: C02, C15, C22.

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